Rogue Wave Solutions to Integrable System by Darboux Transformation

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Xin Kou

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Thesis Examination Committee:

Advisor
Jianke Yang, Ph.D.

Daniel Bentil, Ph.D.

Chairperson
Dennis Clougherty, Ph.D.

Dean, Graduate College
Cynthia J. Forehand, Ph.D.

Date: August 28, 2014

Abstract

The Darboux transformation is one of the main techniques for finding solutions of integrable equations. The Darboux transformation is not only powerful in the construction of muilti-soliton solutions, recently, it is found that the Darboux transformation, after some modification, is also effective in generating the rogue wave solutions. In this thesis, we derive the rogue wave solutions for the Davey-Stewartson-II (DS-II) equation in terms of Darboux transformation. By taking the spectral function as the product of plane wave and rational function, we get the fundamental rogue wave solution and multi-rogue wave solutions via the normal Darboux transformation. Last but not least, we construct a generalized Darboux transformation for DS-II equation by using the limit process. As applications, we use the generalized Darboux transformation to derive the second-order rogue waves. In addition, an alternative way is applied to derive the N-fold Darboux transformation for the nonlinear Schrödinger (NLS) equation. One advantage of this method is that the proof for N-fold Darboux transformation is very straightforward.

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1 Introduction

1.1 History

1.1.1 Rogue Waves

Rogue waves are relatively large and spontaneous ocean surface waves [1]. They were also observed in nonlinear optics [2, 3], water wave tanks [4], atmosphere [5], plasma physics [6] and matter waves (Bose-Einstein condensate) [7].

The very first model for rogue waves is the Peregrine soliton to the well-known focusing nonlinear Schrödinger (NLS) equation,

$$iq_t + q_{xx} + 2|q|^2 q = 0, \tag{1}$$

which is a solution localizing in both space and time and with one dominant peak [8]. Later, this solution was rediscovered by Akhmediv [9], he showed that Peregrine soliton is a limiting case of an Akhmediev breather when time approaches infinity. Such solution demonstrates the essential characteristic of the rogue wave solution, i.e., appear from nowhere and disappear without a trace [10]. The higher-order rogue wave solutions to the focusing NLS equation (1) were also constructed by different techniques, such as Darboux transformation [11, 12], bilinear method [13] and so on. The higher-order rogue wave can be considered as the nonlinear superposition of the fundamental (first-order) rogue wave solutions.

In addition to the focusing NLS equation, many other integrable systems admit rogue waves. For instance, the derivative NLS equation [14, 15], the Hirota equation [16, 17], NLS-Maxwel-Bloch equations [18], Davey-Stewartson-I, II (DS-I,II) [19, 20]. Recently, important progress has been presented in the rogue waves for coupled systems, such as the coupled NLS equations [21, 22], the coupled Hirota equation [23] and so on. These results show the abundant pattern for rogue waves in the multicomponent coupled systems.

1.1.2 Darboux Transformation

The Darboux transformation, first proposed by the French mathematician Gaston Darboux in 1882 on his study of the linear Sturm-Liouville problem[24], is one of the main techniques for finding solutions of integrable equations [25]. Darboux transformation transforms the Lax pair of the nonlinear equation with the old potential to a similar one with new potential, in other words, the Lax pair is covariant with respect to the action of Darboux transformation. Thus one obtains new potential from the old potential.

When we apply Darboux transformation on the trivial solution once, we will get the single soliton and the corresponding Darboux transformation is called the one-fold Darboux transformation. Darboux transformation can be iterated N times, and when the iteration of Darboux transformation acts on the trivial solution, the multisoliton solutions will be generated. Likewise, the iteration of Darboux transformation is named as N- fold Darboux transformation or N-times repeated Darboux transformation.

Darboux transformation, with some modification, is also powerful in generating the rational solutions of integrable equations. The generalized Darboux transformation, which was first introduced by Matveev [26], was applied to calculate the positon solutions of the Korteweg-de Vries (KdV) equation. In recent years, Matveev's generalized Daboux transformation was re-examined to construct the higher-order rogue wave for NLS equation [11], derivative NLS equation[27],etc. It is believed that the generalized Darboux transformation is rather general and can be easily applied to other rogue wave models.

1.2 Motivation

Motivated by the power of the generalized Darboux transformation, we would like to apply this method to the physically interested model. We choose the DS-II equation for the following reasons: first of all, the two-dimensional models can well describe the ocean surface waves. Secondly, DS-II equation has been studied by the bilinear method in [20]. Thus investigating DS-II equation with the Darboux transformation enables us to compare those two methods in finding rogue wave solutions. We are also interested in whether Darboux transformation can provide us with some new rogue wave patterns or not.

When learning the Darboux transformation for the focusing NLS equation, we found the one given in Matveev's book [25] lacked the proof for N-fold Darboux transformation, and we tried to finish the proof but it is not easy. Therefore we turned to another way to derive the N-fold Darboux transformation for NLS equation, which turned out to be very straightforward when it comes to the proof.

1.3 Outline of the Thesis

This thesis is organized as follows.

In section 2, the rogue wave solutions for the DS-II equation are derived in terms of Darboux transformation. Mathematically, the rogue waves are always represented by the rational function. Thus we take the spectral function as the product of plane wave and rational function, then we get the fundamental rogue wave solution and multi-rogue wave solutions via the normal Darboux transformation. Besides, we construct a generalized Darboux transformation for DS-II equation by using the limit process. As applications, we use the generalized Darboux transformation to calculate the second-order rogue waves.

In section 3, an alternative way is applied to derive the N-fold Darboux transformation for the NLS equation. One advantage of this method is that the proof for N-fold Darboux transformation is very straightforward. we also show the equivalence of those two Darboux transformations at the end of this section.

2 Darboux Transformation and Rogue Waves for DS-II Equation

In this section we derive the rogue wave solutions for the DS-II equation in terms of Darboux transformation. Mathematically, the rogue waves are always represented by the rational functions. Thus, by taking the spectral function as the product of plane wave and rational function, we get the fundamental rogue wave solution and multi-rogue wave solutions via the standard Darboux transformation. In addition, we construct a generalized Darboux transformation for DS-II equation by using the limit process. As applications, we use the generalized Darboux transformation to derive the second-order rogue waves.

We begin with the so-called extended Davey-Stewartson-II (EDS-II) system [25]:

$$\begin{aligned} iu_t &= -u_{xx} + u_{yy} - uQ, && (2a)\\ iv_t &= v_{xx} - v_{yy} + vQ, && (2b)\\ w_y &= -Q_x - 2(uv)_x, && (2c)\\ w_x &= Q_y - 2(uv)_y && (2d) \end{aligned}$$

Under the constraints

$$v = \kappa u^*, \qquad \mathrm{Im}\ Q = 0, \tag{3}$$

and the change of variables

$$Q = 2\kappa |u|^2 + S, \tag{4}$$

the EDS-II system (2) reduces to the DS-II equation

$$
\begin{aligned}
& iu_t + u_{xx} - u_{yy} + (2\kappa|u|^2 + S)u = 0, && \text{(5a)}\\
& S_{xx} + S_{yy} = -4\kappa(|u|^2)_{xx}, && \text{(5b)}\\
& \kappa = \pm 1. &&
\end{aligned}
$$

The EDS-II system (2) is the compatibility condition of the Lax pair

$$
\begin{aligned}
\Psi_y &= J\Psi_x + U\Psi, && \text{(6a)}\\
\Psi_t &= 2J\Psi_{xx} + 2U\Psi_x + V\Psi. && \text{(6b)}
\end{aligned}
$$

Here

$$
J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad U = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad V = \begin{pmatrix} (w+iQ)/2 & u_x - iu_y \\ v_x + iv_y & (w-iQ)/2 \end{pmatrix}. \tag{7}
$$

2.1 One-fold Darboux Transformation and Fundamental Rogue Wave Solution

The Lax pair (6) is covariant under the action of one-fold DT [25]:

$$
\begin{aligned}
& \Psi \to \Psi[1] = D[1]\Psi = \Psi_x - \sigma\Psi, && \text{(8a)}\\
& U \to U[1] = U + [J, \sigma], && \text{(8b)}\\
& V \to V[1] = V + 2(\sigma_y + J\sigma_x), && \text{(8c)}
\end{aligned}
$$

where

$$
\sigma = \Psi_{1x}\Psi_1^{-1}, \tag{9}
$$

and $\Psi_1 = \begin{pmatrix} \psi_1 & \psi_2 \\ \phi_1 & \phi_2 \end{pmatrix}$ is a fixed matrix solution of the Lax Pair (6) corresponding to the starting matrix potentials U and V. Additionally, $U[1]$ and $V[1]$ produce new solutions of EDS-II system. This fact is summarized in Theorem 2 [25]. The proof is missing in [25], so that we provide the proof below.

Theorem 1. *The function* $\Psi[1]$ *satisfies the equation*

$$\Psi[1]_y = J\Psi[1]_x + U[1]\Psi, \tag{10a}$$

$$\Psi[1]_t = 2J\Psi[1]_{xx} + 2U[1]\Psi[1]_x + V[1]\Psi[1]. \tag{10b}$$

where $\Psi[1], U[1]$ *and* $V[1]$ *are defined as in* (8).

Proof. Substituting the definition of $\Psi[1]$ (8a) into (37a) and considering the spectral problem (6a), we have

$$J\Psi_{xx} + (U - \sigma J)\Psi_x + (U_x - \sigma_y - \sigma U)\Psi = J\Psi_{xx} + (U[1] - J\sigma)\Psi_x + (-J\sigma_x - U[1]\sigma)\Psi \tag{11}$$

Comparing the coefficients of Ψ_x and Ψ we obtain:

$$\Psi_x : \quad U[1] = U + [J, \sigma], \tag{12a}$$

$$\Psi : \quad U_x - \sigma_y - \sigma U + J\sigma_x + U[1]\sigma = 0, \tag{12b}$$

where the commutator of square matrix A and B is defined as $[A, B] = AB - BA$. It is obvious that (12a) is just the transformed potential matrix given by (8b).

Substitution of (12a) into (12b) yields the relation

$$J\sigma_x - \sigma_y = -[U, \sigma] - [J, \sigma]\sigma - U_x, \tag{13}$$

which holds due to the definition of σ. In fact,

$$J\sigma_x = J(\Psi_{1x}\Psi_1^{-1})_x = J\Psi_{1xx}\Psi_1^{-1} - J\Psi_{1x}(\Psi_1^{-1}\Psi_{1x}\Psi_1^{-1}) = J\Psi_{1xx}\Psi_1^{-1} - J\sigma^2, \tag{14}$$

and

$$\begin{aligned}
\sigma_y &= \Psi_{1xy}\Psi_1^{-1} + \Psi_{1x}(\Psi_1^{-1})_y \\
&= \Psi_{1yx}\Psi_1^{-1} - \Psi_{1x}(\Psi_1^{-1}\Psi_{1y}\Psi_1^{-1}) \\
&= (J\Psi_{1xx} + U_x\Psi_1 + U\Psi_{1x})\Psi_1^{-1} - \sigma(J\Psi_{1x} + U\Psi_1)\Psi_1^{-1} \\
&= J\Psi_{1xx}\Psi_1^{-1} + U_x + U\sigma - \sigma J\sigma - \sigma U \\
&= J\Psi_{1xx}\Psi_1^{-1} + U_x + [U, \sigma] - \sigma J\sigma,
\end{aligned} \tag{15}$$

then

$$J\sigma_x - \sigma_y = -[U, \sigma] - J\sigma^2 + \sigma J\sigma - U_x = -[U, \sigma] - [J, \sigma]\sigma - U_x. \tag{16}$$

Next we verify (37b). Similarly, substituting (8a) into (37b), we get

$$\begin{aligned}
\text{LHS of 7(b)} &= \Psi_{xt} - \sigma_t\Psi - \sigma\Psi_t \\
&= 2J\Psi_{xxx} + 2U_x\Psi_x + 2U\Psi_{xx} + V_x\Psi + V\Psi_x - \sigma_t\Psi - 2\sigma J\Psi_{xx} - 2\sigma U\Psi_x - \sigma V\Psi \\
&= 2J\Psi_{xxx} + (2U - 2\sigma J)\Psi_{xx} + (2U_x + V - 2\sigma U)\Psi_x + (V_x - \sigma_t - \sigma V)\Psi,
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\text{RHS of 7(b)} &= 2J(\Psi_x - \sigma\Psi)_{xx} + 2U[1](\Psi_x - \sigma\Psi)_x + V[1](\Psi_x - \sigma\Psi) \\
&= 2J\Psi_{xxx} - 2J(\sigma_{xx}\Psi + 2\sigma_x\Psi_x + \sigma\Psi_{xx}) + 2U[1](\Psi_{xx} - \sigma_x\Psi - \sigma\Psi_x) + V[1](\Psi_x - \sigma\Psi) \\
&= 2J\Psi_{xxx} + (2U[1] - 2J\sigma)\Psi_{xx} + (V[1] - 4J\sigma_x - 2U[1]\sigma)\Psi_x \\
&- (2J\sigma_{xx} + 2U[1]\sigma_x + V[1]\sigma)\Psi.
\end{aligned}$$

Comparing the coefficients of Ψ_{xxx}, Ψ_{xx}, Ψ_x and Ψ:

$$\Psi_{xxx}: \qquad 2J = 2J, \tag{18a}$$

$$\Psi_{xx}: \qquad U[1] = U + [J, \sigma] \tag{18b}$$

$$\Psi_x: \qquad 2U_x + V - 2\sigma U = V[1] - 4J\sigma_x - 2U[1]\sigma \tag{18c}$$

$$\Psi: \qquad V_x - \sigma_t - \sigma V = -(2J\sigma_{xx} + 2U[1]\sigma_x + V[1]\sigma) \tag{18d}$$

Substituting (18b) into (18c), we have

$$V[1] = V + 2(U_x + [U, \sigma] + [J, \sigma]\sigma) + 4J\sigma_x. \tag{19}$$

Now from (13) and (19) we obtain

$$V[1] = V + 2(\sigma_y + J\sigma_x). \tag{20}$$

Substitution of (18b) and (20) into (18d) produces:

$$\sigma_t - 2J\sigma_{xx} - 2[J, \sigma]\sigma_x - 2(\sigma_y + J\sigma_x)\sigma - V_x - [V, \sigma] - 2U\sigma_x = 0. \tag{21}$$

We will verify (21) in the following by the definition of σ.

$$\begin{aligned}
\sigma_t &= (\Psi_{1x}\Psi_1^{-1})_t = \Psi_{1xt}\Psi_1^{-1} + \Psi_{1x}(\Psi_1^{-1})_t \\
&= \Psi_{1tx}\Psi_1^{-1} - \Psi_{1x}(\Psi_1^{-1}\Psi_{1t}\Psi_1^{-1}) \\
&= 2J\Psi_{1xxx}\Psi_1^{-1} + 2(U_x\Psi_{1x} + U\Psi_{1xx})\Psi_1^{-1} + V_x + V\Psi_{1x}\Psi_1^{-1} - \sigma(2J\Psi_{1xx} + 2U\Psi_{1x} + V\Psi_1)\Psi_1^{-1} \\
&= 2J\Psi_{1xxx}\Psi_1^{-1} + 2U_x\sigma + 2U\Psi_{1xx}\Psi_1^{-1} + V_x + [V, \sigma] - 2\sigma J\Psi_{1xx}\Psi_1^{-1} - 2\sigma U\sigma \\
&= V_x + [V, \sigma] + 2J\Psi_{1xxx}\Psi_1^{-1} + 2(U\Psi_{1xx}\Psi_1^{-1} - \sigma J\Psi_{1xx}\Psi_1^{-1}) + 2(U_x\sigma - \sigma U\sigma),
\end{aligned} \tag{22}$$

$$-2J\sigma_{xx} = -2J(\Psi_{1xx}\Psi_1^{-1} - \sigma^2)_x = -2J(\Psi_{1xxx}\Psi_1^{-1} - \Psi_{1xx}\Psi_1^{-1}\sigma + 2J\sigma\sigma_x + 2J\sigma_x\sigma). \quad (23)$$

Then

$$\text{LHS of (21)} = 2(J\sigma_x\sigma + \sigma J\sigma_x + \sigma J\sigma^2 + J\sigma^3 - \sigma J\Psi_{1xx}\Psi_1^{-1} - J\Psi_{1xx}\Psi_1^{-1}\sigma) \quad (24)$$

Since $J\Psi_{1xx}\Psi_1^{-1} = J\sigma_x + J\sigma^2$, then the last two terms in the above equation become $-J\sigma_x\sigma - \sigma J\sigma_x - \sigma J\sigma^2 - J\sigma^3$, which are just the opposite of the first four terms. It indicates that (21) holds. □

According to (8b) and (8b), the new solutions obtained from one-fold Darboux transformation for EDS-II system are given by

$$\begin{aligned} u[1] &= u + 2i\sigma_{12} = u + 2i\frac{\psi_1\psi_{2x} - \psi_2\psi_{1x}}{\psi_1\phi_2 - \phi_1\psi_2}, && (25a)\\ v[1] &= v - 2i\sigma_{21} = v + 2i\frac{\phi_1\phi_{2x} - \phi_2\phi_{1x}}{\psi_1\phi_2 - \phi_1\psi_2}, && (25b)\\ Q[1] &= Q + 2\partial_x(\sigma_{11} + \sigma_{22}) - 2i\partial_y(\sigma_{11} - \sigma_{22}), && (25c) \end{aligned}$$

where

$$\Psi_1 = \begin{pmatrix} \psi_1 & \psi_2 \\ \phi_1 & \phi_2 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}. \quad (26)$$

In order to find the new solution for DS-II equation, we require that the new solutions for EDS-II system satisfy the constraints (3) as well, that is

$$v[1] = \kappa u[1]^*, \qquad \text{Im } Q[1] = 0. \quad (27)$$

Then one possible choice of the starting solution of Lax pair is

$$\Psi_1 = \begin{pmatrix} \psi_1 & \phi_1^* \\ \phi_1 & \kappa\psi_1^* \end{pmatrix}. \tag{28}$$

In other words, $(\psi_2, \phi_2)^T$ will be replaced with $(\phi_1^*, \kappa\psi_1^*)^T$. Then the new solutions $u[1]$ and $Q[1]$ for DS-II equation are given by

$$u[1] = u + 2i\frac{\psi_1\phi_{1x}^* - \phi_1^*\psi_{1x}}{\Delta} = u + 2i\frac{\begin{vmatrix} \psi_1 & \phi_1^* \\ \psi_{1x} & \phi_{1x}^* \end{vmatrix}}{\begin{vmatrix} \psi_1 & \phi_1^* \\ \phi_1 & \kappa\psi_1^* \end{vmatrix}}, \tag{29a}$$

$$Q[1] = Q + 2\partial_x(\sigma_{11} + \sigma_{22}) - 2i\partial_y(\sigma_{11} - \sigma_{22}), \tag{29b}$$

$$\Delta = \kappa|\psi_1|^2 - |\phi_1|^2, \quad \sigma_{11} = \frac{\kappa\psi_1^*\psi_{1x} - \phi_1\phi_{1x}^*}{\Delta}, \quad \sigma_{22} = \frac{\kappa\psi_1\psi_{1x}^* - \phi_1^*\phi_{1x}}{\Delta}.$$

When seed solution $u = 0$ and taking $Q = 0$, we can get nonsingular line soliton solutions for DS-II equation in the case $\kappa = -1$, which can be found in the literature [31].

When $\kappa = 1$, no rogue wave solutions can be expected due to the modulation stability of constant background solution [20, 32]. Thus we only consider $\kappa = -1$ throughout the rest of the thesis.

To derive the rogue wave solutions for $\kappa = -1$, without loss of generality, we start with the seed solution $u = 1$, then Q is forced to be zero. A key point here is to determine the expression for the solution of Lax pair. Since we know rogue waves are rational functions mathematically, we simply assume the solution for the Lax pair as the product of plain wave and rational function and determine the coefficients by substituting the solution into the Lax pair. Then we come to the corresponding solution for the Lax pair:

$$\begin{pmatrix} \psi_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} e^{i\beta_1/2} e^{ix\sin\beta_1 + iy\cos\beta_1 - t\sin 2\beta_1} (ix\cos\beta_1 - iy\sin\beta_1 - 2t\cos 2\beta_1 + i/2 + c_{11} + ic_{12}) \\ ie^{-i\beta_1/2} e^{ix\sin\beta_1 + iy\cos\beta_1 - t\sin 2\beta_1} (ix\cos\beta_1 - iy\sin\beta_1 - 2t\cos 2\beta_1 - i/2 + c_{11} + ic_{12}) \end{pmatrix}, \tag{30}$$

where β_1, c_{11} and c_{12} are all real parameters. We recognize that this solution can be written as the partial derivative respect to β_1:

$$\begin{pmatrix} \psi_1 \\ \phi_1 \end{pmatrix} = (\partial_{\beta_1} + c_{11} + ic_{12}) \begin{pmatrix} e^{i\beta_1/2} \\ ie^{-i\beta_1/2} \end{pmatrix} e^{ix\sin\beta_1 + iy\cos\beta_1 - t\sin 2\beta_1}. \tag{31}$$

Substitution the above equation (31) into (29) yields the fundamental rogue wave solution:

$$u[1] = e^{2i\beta_1}[1 - \frac{4 + 16it\cos 2\beta_1 - 8ic_{11}}{1 + 4(x\cos\beta_1 - y\sin\beta_1 + c_{12})^2 + (4t\cos 2\beta_1 - 2c_{11})^2}], \tag{32a}$$

$$Q[1] = 16\cos 2\beta_1 \frac{1 - 4(x\cos\beta_1 - y\sin\beta_1 + c_{12})^2 + (4t\cos 2\beta_1 - 2c_{11})^2}{[1 + 4(x\cos\beta_1 - y\sin\beta_1 + c_{12})^2 + (4t\cos 2\beta_1 - 2c_{11})^2]^2}, \tag{32b}$$

After a shift of spatial and temporal variables,

$$x \to x - \frac{c_{12}}{2\cos\beta_1}, \quad y \to y + \frac{c_{12}}{2\sin\beta_1}, \quad t \to t + \frac{c_{11}}{2\cos 2\beta_1}, \tag{33}$$

parameters c_{11} and c_{12} can be removed. Then this fundamental rogue wave solution can be written as

$$
\begin{aligned}
u[1] &= e^{2i\beta_1}\left[1-\frac{4+16it\cos 2\beta_1}{1+4(x\cos\beta_1-y\sin\beta_1)^2+16t^2\cos^2 2\beta_1}\right], && \text{(34a)}\\
Q[1] &= 16\cos 2\beta_1\frac{1-4(x\cos\beta_1-y\sin\beta_1)^2+16t^2\cos^2 2\beta_1}{[1+4(x\cos\beta_1-y\sin\beta_1)^2+16t^2\cos^2 2\beta_1]^2}, && \text{(34b)}\\
\beta_1 &\neq \pm\pi/4.
\end{aligned}
$$

The expression for $S[1]$ can be obtained by (4). It is obvious that β_1 is the only real parameter. The solution we get here coincides with the one in [20]. The solution $u[1]$ describes a line wave, of which the width is constant for any β_1 value. When $t \to \pm\infty$, the solution u[1] uniformly approached the constant background 1, and at time $t = 0$, the amplitude of $u[1]$ attains three times the background amplitude at the center $x\cos\beta_1 - y\sin\beta_1 = 0$ of the line wave, and this agrees with the description for rogue wave, which is "appear from nowhere and disappear without a trace". The rouge wave solution $u[1]$ is illustrated in figure 1 with $\beta_1 = \pi/6$.

When $\beta_1 = \pm\pi/4$, i.e. $\cos 2\beta_1 = 0$, solution (32) is not the rogue wave any more. In fact, (32) becomes

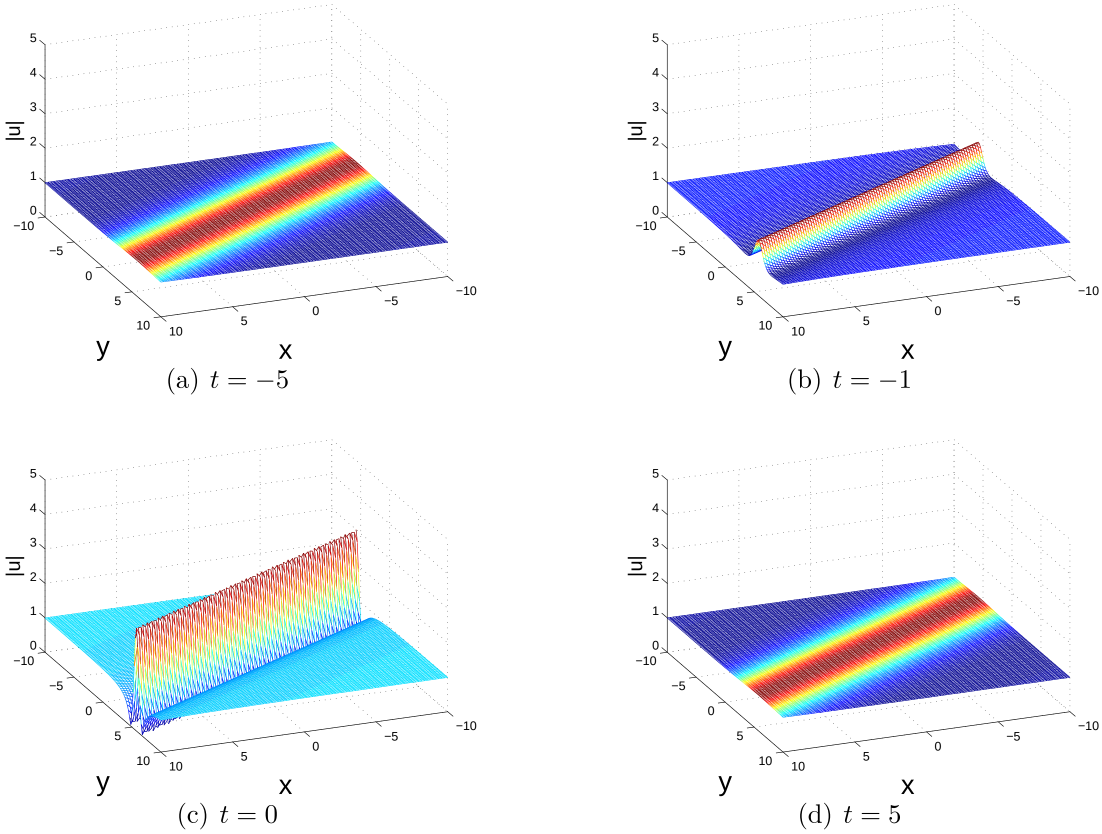
$$
u[1] = i\left[1-\frac{4-8ic_{11}}{1+(\sqrt{2}x\pm\sqrt{2}y)^2+4c_{11}^2}\right], \tag{35}
$$

Here we have eliminated the parameter c_{12}. It is evident that (35) represents a stationary line soliton solution sitting on the constant background.

2.2 N-fold Darboux Transformation

We can iterate the one-fold Darboux transformation N times to get the N-fold Darboux transformation, and it can be given by [25]

$$
\Psi[N] = D[N]\Psi = \partial_x^N\Psi - (s_1\partial_x^{N-1}\Psi + \ldots + s_N\Psi), \tag{36}
$$



(a) $t = -5$ (b) $t = -1$

(c) $t = 0$ (d) $t = 5$

Figure 1: Fundamental rogue wave (34) *with* $\beta_1 = \pi/6$

where coefficients $s_k (k = 1, \ldots, N)$ are 2×2 matrices. $\Psi[N]$ is supposed to satisfy the Lax pair (6) with the transformed potential matrix $U[N]$ and $V[N]$:

$$\begin{aligned} \Psi[N]_y &= J\Psi[N]_x + U[N]\Psi[N], & \text{(37a)} \\ \Psi[N]_t &= 2J\Psi[N]_{xx} + 2U[N]\Psi[N]_x + V[N]\Psi[N], & \text{(37b)} \end{aligned}$$

where

$$U[N] = \begin{pmatrix} 0 & u[N] \\ v[N] & 0 \end{pmatrix}, \quad V[N] = \begin{pmatrix} (w[N] + iQ[N])/2 & u[N]_x - iu[N]_y \\ v[N]_x + iv[N]_y & (w[N] - iQ[N])/2 \end{pmatrix}. \quad (38)$$

Substituting (36) into (37a) and comparing the coefficients of $\partial_x^N \Psi$ and $\partial_x^{N-1}\Psi$, we get

$$\begin{aligned} \partial_x^N \Psi : \quad U[N] &= U + [J, s_1], & \text{(39a)} \\ \partial_x^{N-1}\Psi : \quad [s_2, J] &= Js_{1x} - s_{1y} + U[N]s_1 - s_1 U + NU_x. & \text{(39b)} \end{aligned}$$

Similarly, Substituting (36) into (37b) and comparing the coefficients of $\partial_x^N \Psi$, we have

$$V[N] = V + 2NU_x + 4Js_{1x} + 2U[N]s_1 - 2s_1 U - 2[s_2, J]. \quad (40)$$

Considering (39b), we have

$$V[N] = V + 2(s_{1y} + Js_{1x}). \quad (41)$$

The coefficients $s_k (k = 1, \ldots, N)$ can be determined by the conditions:

$$D[N]\Psi_k = 0, \qquad k = 1, \ldots, N, \quad (42)$$

where $\Psi_k = \begin{pmatrix} \psi_{2k-1} & \psi_{2k} \\ \phi_{2k-1} & \phi_{2k} \end{pmatrix}$ are some fixed solutions of linear problem (6). The above conditions can be written as the system of linear matrix equations

$$
\begin{aligned}
&(s_1, s_2, \ldots, s_N)\Delta = (\partial_x^N \Psi_1, \partial_x^N \Psi_2, \ldots, \partial_x^N \Psi_N), \\
&\Delta = \begin{pmatrix} \partial_x^{N-1}\Psi_1 & \partial_x^{N-1}\Psi_2 & \ldots & \partial_x^{N-1}\Psi_N \\ \partial_x^{N-2}\Psi_1 & \partial_x^{N-2}\Psi_2 & \ldots & \partial_x^{N-2}\Psi_N \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_1 & \Psi_2 & \ldots & \Psi_N \end{pmatrix}.
\end{aligned} \tag{43}
$$

By solving (43), we can obtain the explicit representations of s_k. In particular,

$$
s_1 = \frac{\sum_{k=1}^N \partial_x^N \Psi_k \Delta_{k1}}{\det \Delta}, \quad \Delta_{k1} = \begin{pmatrix} \delta^*_{1,2k-1} & \delta^*_{2,2k-1} \\ \delta^*_{1,2k} & \delta^*_{2,2k} \end{pmatrix}, \tag{44}
$$

where δ_{ij} is the entry in the i-th row and j-th column of matrix Δ and δ^*_{ij} is the cofactor of δ_{ij}.

Setting

$$
s_1 = \begin{pmatrix} s_1^{(11)} & s_1^{(12)} \\ s_1^{(21)} & s_1^{(22)} \end{pmatrix}, \tag{45}
$$

and recalling (39), we arrive at

$$
\begin{aligned}
u[N] &= u + 2is_1^{(12)}, && (46a)\\
v[N] &= v - 2is_1^{(21)}, && (46b)\\
Q[N] &= Q + 2\partial_x(s_1^{(11)} + s_1^{(22)}) - 2i\partial_y(s_1^{(11)} + s_1^{(22)}), && (46c)\\
s_1^{(11)} &= \Sigma_{11}/\Sigma,\ s_1^{(12)} = \Sigma_{12}/\Sigma,\ s_1^{(21)} = \Sigma_{21}/\Sigma,\ s_1^{(22)} = \Sigma_{22}/\Sigma,
\end{aligned}
$$

where

$$\Sigma = \begin{vmatrix} \partial_x^{N-1}\psi_1 & \partial_x^{N-1}\psi_2 & \dots & \partial_x^{N-1}\psi_{2N} \\ \partial_x^{N-1}\phi_1 & \partial_x^{N-1}\phi_2 & \dots & \partial_x^{N-1}\phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1 & \psi_2 & \dots & \psi_{2N} \\ \phi_1 & \phi_2 & \dots & \phi_{2N} \end{vmatrix}, \tag{47}$$

$\Sigma_{1j}(j = 1, 2)$ are formed by replacing j-th row of Σ with $(\partial_x^N \psi_1, \partial_x^N \psi_2, \dots, \partial_x^N \psi_{2N})$ and $\Sigma_{2j}(j = 1, 2)$ are formed by replacing j-th row of Σ with $(\partial_x^N \phi_1, \partial_x^N \phi_2, \dots, \partial_x^N \phi_{2N})$.

The reduction requirement $v[N] = -u[N]^*$ can be realized when $\psi_{2k} = \phi_{2k-1}^*$ and $\phi_{2k} = -\psi_{2k-1}^*$. In other words,

$$\Psi_k = \begin{pmatrix} \psi_k & \phi_k^* \\ \phi_k & -\psi_k^* \end{pmatrix}. \tag{48}$$

Taking into account (71) and rearranging the determinants in (46a), we arrive at the final determinant representation of transformed solution under N-fold Darboux transformation:

$$u[N]=u+2i\frac{\begin{vmatrix} \partial_x^{N-1}\psi_1 & \dots & \partial_x^{N-1}\psi_N & \partial_x^{N-1}\phi_1^* & \dots & \partial_x^{N-1}\phi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1 & \dots & \psi_N & \phi_1^* & \dots & \phi_N^* \\ \partial_x^{N}\psi_1 & \dots & \partial_x^{N}\psi_N & \partial_x^{N}\phi_1^* & \dots & \partial_x^{N}\phi_N^* \\ \partial_x^{N-2}\phi_1 & \dots & \partial_x^{N-2}\phi_N & -\partial_x^{N-2}\psi_1^* & \dots & -\partial_x^{N-2}\psi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_1 & \dots & \phi_N & -\psi_1^* & \dots & -\psi_N^* \end{vmatrix}}{\begin{vmatrix} \partial_x^{N-1}\psi_1 & \dots & \partial_x^{N-1}\psi_N & \partial_x^{N-1}\phi_1^* & \dots & \partial_x^{N-1}\phi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1 & \dots & \psi_N & \phi_1^* & \dots & \phi_N^* \\ \partial_x^{N-1}\phi_1 & \dots & \partial_x^{N-1}\phi_N & -\partial_x^{N-1}\psi_1^* & \dots & -\partial_x^{N-1}\psi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_1 & \dots & \phi_N & -\psi_1^* & \dots & -\psi_N^* \end{vmatrix}}. \tag{49}$$

and

$$Q[N]=Q+2\partial_x(s_1^{(11)}+s_1^{(22)})-2i\partial_y(s_1^{(11)}+s_1^{(22)}), \tag{50}$$

$$s_1^{(11)} = \frac{\begin{vmatrix} \partial_x^N\psi_1 & \dots & \partial_x^N\psi_N & \partial_x^N\phi_1^* & \dots & \partial_x^N\phi_N^* \\ \partial_x^{N-2}\psi_1 & \dots & \partial_x^{N-2}\psi_N & \partial_x^{N-2}\phi_1^* & \dots & \partial_x^{N-2}\phi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1 & \dots & \psi_N & \phi_1^* & \dots & \phi_N^* \\ \partial_x^{N-1}\phi_1 & \dots & \partial_x^{N-1}\phi_N & -\partial_x^{N-1}\psi_1^* & \dots & -\partial_x^{N-1}\psi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_1 & \dots & \phi_N & -\psi_1^* & \dots & -\psi_N^* \end{vmatrix}}{\begin{vmatrix} \partial_x^{N-1}\psi_1 & \dots & \partial_x^{N-1}\psi_N & \partial_x^{N-1}\phi_1^* & \dots & \partial_x^{N-1}\phi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1 & \dots & \psi_N & \phi_1^* & \dots & \phi_N^* \\ \partial_x^{N-1}\phi_1 & \dots & \partial_x^{N-1}\phi_N & -\partial_x^{N-1}\psi_1^* & \dots & -\partial_x^{N-1}\psi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_1 & \dots & \phi_N & -\psi_1^* & \dots & -\psi_N^* \end{vmatrix}},$$

$$s_1^{(22)} = \frac{\begin{vmatrix} \partial_x^{N-1}\psi_1 & \dots & \partial_x^{N-1}\psi_N & \partial_x^{N-1}\phi_1^* & \dots & \partial_x^{N-1}\phi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1 & \dots & \psi_N & \phi_1^* & \dots & \phi_N^* \\ \partial_x^{N}\phi_1 & \dots & \partial_x^{N}\phi_N & -\partial_x^{N}\psi_1^* & \dots & -\partial_x^{N}\psi_N^* \\ \partial_x^{N-2}\phi_1 & \dots & \partial_x^{N-2}\phi_N & -\partial_x^{N-2}\psi_1^* & \dots & -\partial_x^{N-2}\psi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_1 & \dots & \phi_N & -\psi_1^* & \dots & -\psi_N^* \end{vmatrix}}{\begin{vmatrix} \partial_x^{N-1}\psi_1 & \dots & \partial_x^{N-1}\psi_N & \partial_x^{N-1}\phi_1^* & \dots & \partial_x^{N-1}\phi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1 & \dots & \psi_N & \phi_1^* & \dots & \phi_N^* \\ \partial_x^{N-1}\phi_1 & \dots & \partial_x^{N-1}\phi_N & -\partial_x^{N-1}\psi_1^* & \dots & -\partial_x^{N-1}\psi_N^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_1 & \dots & \phi_N & -\psi_1^* & \dots & -\psi_N^* \end{vmatrix}}.$$

There are three types of non-fundamental rogue wave solutions reported in [20], which are multi-rogue waves, higher-order rogue waves and exploding rogue waves. We will first consider the multi-rogue waves.

2.2.1 Multi-rogue Wave Solutions

Multi-rogue wave solutions describe the interaction between fundamental rogue waves and they can be obtained from (49) by taking $u = 1$ and

$$\begin{pmatrix} \psi_j \\ \phi_j \end{pmatrix} = (\partial_{\beta_j} + c_{j1} + ic_{j2}) \begin{pmatrix} e^{i\beta_j/2} \\ ie^{-i\beta_j/2} \end{pmatrix} e^{ix\sin\beta_j + iy\cos\beta_j - t\sin 2\beta_j}, \quad 1 \leq j \leq N, \tag{51}$$

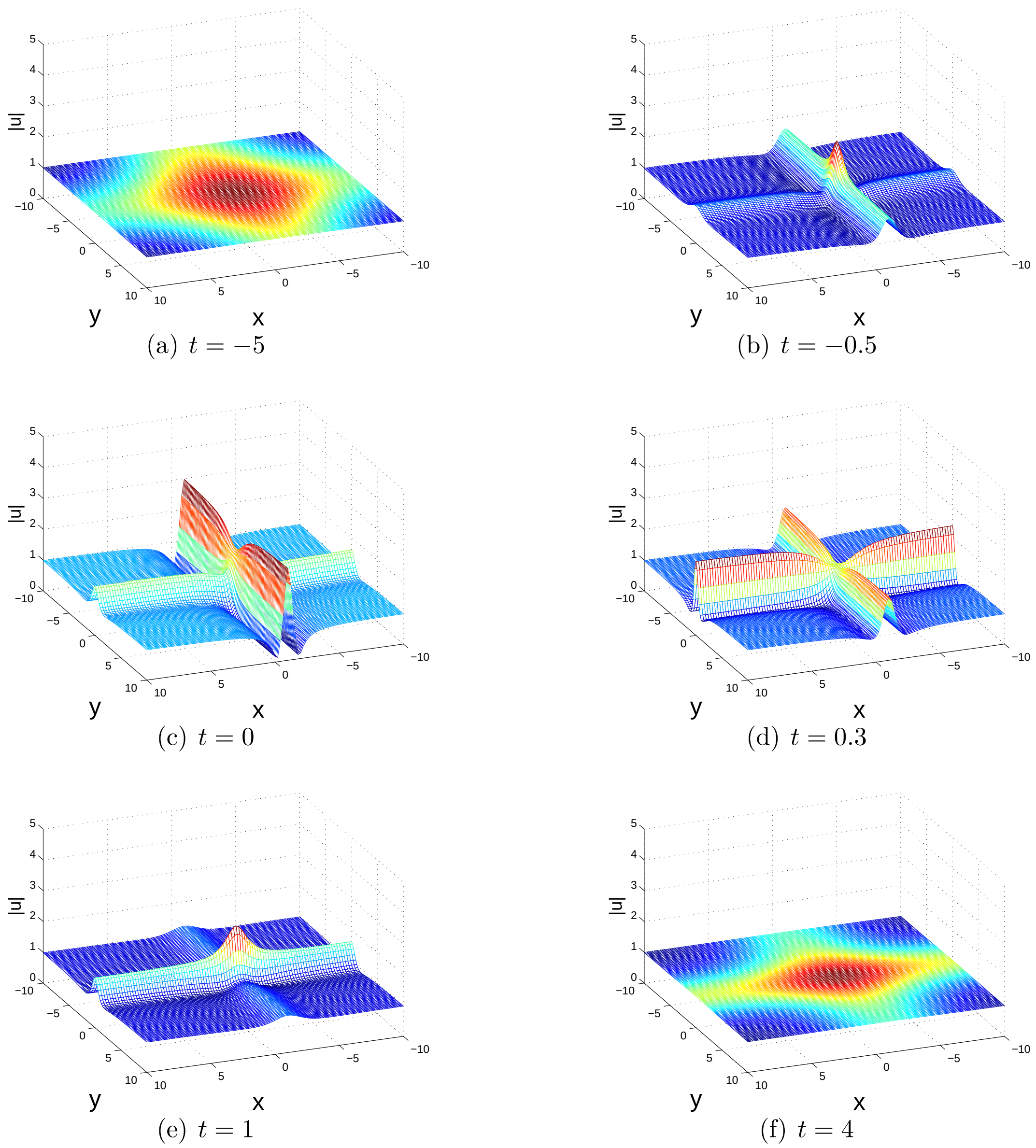
where β_j, c_{j1} and c_{j2} are all real parameters.

We will investigate a two-rogue wave as an example. By taking

$$N = 2,\ \beta_1 = \pi,\ \beta_2 = \pi/2,\ c_{11} = 1,\ c_{12} = 0,\ c_{21} = 0,\ c_{22} = -1, \tag{52}$$

we get the corresponding solution u and $|u|$ is illustrated in the Fig.2. As we can see, the solution uniformly goes to the constant background 1 when $t \to \pm\infty$, and in the intermediate times, the interaction between two fundamental rogue waves occurs.

In the general N-rogue wave solution (49), β_j, c_{j1} and c_{j2} $(1 \leq j \leq N)$ are free real parameters. Among those parameters, three of them can be removed by a shift of the (x, y, t) axes. Therefore we conclude that N-rogue wave solution contains $3(N-1)$ irreducible real parameters. This fits in the conjecture in literature [20].



(a) $t = -5$ (b) $t = -0.5$

(c) $t = 0$ (d) $t = 0.3$

(e) $t = 1$ (f) $t = 4$

Figure 2: A two-rogue wave (49) *with parameters* (52).

2.2.2 Higher-order Rogue Wave Solutions

Another type of non-fundamental rogue wave solutions is the higher-order rogue waves. They can be derived by generalizing the standard Darboux Tranformation(49). Same as before, we first take $N = 2$ as an example to see how second-order rogue waves are generated, then extend to the N case.

In (49),by taking

$$N = 2, \quad u = 1, \tag{53}$$

we get the determinant representation of two-fold Darboux transformation:

$$u[2] = 1 + 2i \frac{\begin{vmatrix} \partial_x\psi_1 & \partial_x\psi_2 & \partial_x\phi_1^* & \partial_x\phi_2^* \\ \psi_1 & \psi_2 & \phi_1^* & \phi_2^* \\ \partial_x^2\psi_1 & \partial_x^2\psi_2 & \partial_x^2\phi_1^* & \partial_x^2\phi_2^* \\ \phi_1 & \phi_2 & -\psi_1^* & -\psi_2^* \end{vmatrix}}{\begin{vmatrix} \partial_x\psi_1 & \partial_x\psi_2 & \partial_x\phi_1^* & \partial_x\phi_2^* \\ \psi_1 & \psi_2 & \phi_1^* & \phi_2^* \\ \partial_x\phi_1 & \partial_x\phi_2 & -\partial_x\psi_1^* & -\partial_x\psi_2^* \\ \phi_1 & \phi_2 & -\psi_1^* & -\psi_2^* \end{vmatrix}}. \tag{54}$$

To get a generalized Darboux transformation, we begin with the assumption that [26, 11]

$$\psi_2 = \psi_1(\beta_1 + \epsilon), \tag{55a}$$

$$\phi_2 = \phi_1(\beta_1 + \epsilon), \tag{55b}$$

where $(\psi_1, \phi_1)^T$ is defined as in (31) and ϵ is a small parameter. Expanding ψ_1 and ϕ_1 at

β_1, we obtain

$$\psi_2 = \psi_1 + \psi_1[1,1]\epsilon + \psi_1[1,2]\epsilon^2 + \ldots + \psi_1[1,k]\epsilon^k + \ldots, \tag{56a}$$

$$\phi_2 = \phi_1 + \phi_1[1,1]\epsilon + \phi_1[1,2]\epsilon^2 + \ldots + \phi_1[1,k]\epsilon^k + \ldots, \tag{56b}$$

where

$$\psi_1[1,k] = \frac{1}{k!}\frac{\partial^k}{\partial \beta_1^k}\psi_1(\beta_1) \tag{57a}$$

$$\phi_1[1,k] = \frac{1}{k!}\frac{\partial^k}{\partial \beta_1^k}\phi_1(\beta_1). \tag{57b}$$

Substituting (57) into (54) and using the properties of determinant, we arrive at

$$u[2] = 1 + 2i \frac{\begin{vmatrix} \partial_x\psi_1 & \epsilon(\partial_x(\psi_1[1,1] + O(\epsilon))) & \partial_x\phi_1^* & \epsilon(\partial_x(\phi_1^*[1,1] + O(\epsilon))) \\ \psi_1 & \epsilon(\psi_1[1,1] + O(\epsilon)) & \phi_1^* & \epsilon(\phi_1^*[1,1] + O(\epsilon)) \\ \partial_x^2\psi_1 & \epsilon(\partial_x^2(\psi_1[1,1] + O(\epsilon))) & \partial_x^2\phi_1^* & \epsilon(\partial_x^2(\phi_1^*[1,1] + O(\epsilon))) \\ \phi_1 & \epsilon(\phi_1[1,1] + O(\epsilon)) & -\psi_1^* & -\epsilon(\psi_1^*[1,1] + O(\epsilon)) \end{vmatrix}}{\begin{vmatrix} \partial_x\psi_1 & \epsilon(\partial_x\psi_1[1,1] + O(\epsilon)) & \partial_x\phi_1 & \epsilon(\partial_x(\phi_1[1,1] + O(\epsilon))) \\ \psi_1 & \epsilon(\psi_1[1,1] + O(\epsilon)) & \phi_1^* & \epsilon(\phi_1^*[1,1] + O(\epsilon)) \\ \partial_x\phi_1 & \epsilon(\partial_x(\phi_1[1,1] + O(\epsilon))) & \partial_x\psi_1^* & -\epsilon(\partial_x(\psi_1^*[1,1] + O(\epsilon))) \\ \phi_1 & \epsilon(\phi_1[1,1] + O(\epsilon)) & -\psi_1^* & -\epsilon(\psi_1^*[1,1] + O(\epsilon)) \end{vmatrix}}$$

$$= 1 + 2i \frac{\begin{vmatrix} \partial_x\psi_1 & \partial_x(\psi_1[1,1] + O(\epsilon)) & \partial_x\phi_1^* & \partial_x(\phi_1^*[1,1] + O(\epsilon)) \\ \psi_1 & \psi_1[1,1] + O(\epsilon) & \phi_1^* & \phi_1^*[1,1] + O(\epsilon) \\ \partial_x^2\psi_1 & \partial_x^2(\psi_1[1,1] + O(\epsilon)) & \partial_x^2\phi_1^* & \partial_x^2(\phi_1^*[1,1] + O(\epsilon)) \\ \phi_1 & \phi_1[1,1] + O(\epsilon) & -\psi_1^* & -\psi_1^*[1,1] + O(\epsilon) \end{vmatrix}}{\begin{vmatrix} \partial_x\psi_1 & \partial_x\psi_1[1,1] + O(\epsilon) & \partial_x\phi_1 & \partial_x(\phi_1[1,1] + O(\epsilon)) \\ \psi_1 & \psi_1[1,1] + O(\epsilon) & \phi_1^* & \phi_1^*[1,1] + O(\epsilon) \\ \partial_x\phi_1 & \partial_x(\phi_1[1,1] + O(\epsilon)) & \partial_x\psi_1^* & -\partial_x(\psi_1^*[1,1] + O(\epsilon)) \\ \phi_1 & \phi_1[1,1] + O(\epsilon) & -\psi_1^* & -\psi_1^*[1,1] + O(\epsilon) \end{vmatrix}} \tag{58}$$

where $\lim_{\epsilon\to 0} O(\epsilon) = 0$. Then taking the limit process $\epsilon \to 0$, we have the generalized two-fold Darboux transformation, i.e. the general expression for the second-order rogue wave solution

$$u[2]=1+2i\frac{\begin{vmatrix}\partial_x\psi_1 & \partial_x\psi_1[1,1] & \partial_x\phi_1^* & \partial_x\phi_1^*[1,1]\\ \psi_1 & \psi_1[1,1] & \phi_1^* & \phi_1^*[1,1]\\ \partial_x^2\psi_1 & \partial_x^2\psi_1[1,1] & \partial_x^2\phi_1^* & \partial_x^2\phi_1^*[1,1]\\ \phi_1 & \phi_1[1,1] & -\psi_1^* & -\psi_1^*[1,1]\end{vmatrix}}{\begin{vmatrix}\partial_x\psi_1 & \partial_x\psi_1[1,1] & \partial_x\phi_1 & \partial_x\phi_1[1,1]\\ \psi_1 & \psi_1[1,1] & \phi_1^* & \phi_1^*[1,1]\\ \partial_x\phi_1 & \partial_x\phi_1[1,1] & -\partial_x\psi_1^* & -\partial_x\psi_1^*[1,1]\\ \phi_1 & \phi_1[1,1] & -\psi_1^* & -\psi_1^*[1,1]\end{vmatrix}}. \tag{59}$$

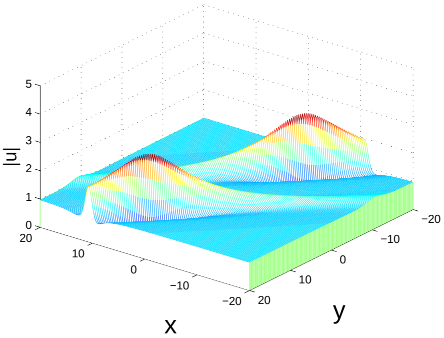
The above second-order rogue wave solution contains three free real parameter β_1, c_{11} and c_{12}, where β_1 plays the crucial role. In fact, when $\beta_1\neq\pm\pi/4$, (59) is not a rogue wave, since the solution does not uniformly go to the constant background. We will not go further on this but focus on the rogue wave solutions, which correspond to $\beta_1=\pm\pi/4$. Different from the multi-rogue waves obtained in the previous section, which "appears from nowhere" and "disappear with no trace", the second order rogue wave we find here, especially for $\beta_1=\pi/4$, goes to the constant background when $t\to\infty$ but does not start from the constant background. The opposite behavior occurs for the solution when $\beta_1=-\pi/4$.

Next we investigate the solution for $\beta_1=\pi/4$ thoroughly by showing its explicit expression. By taking $\beta_1=\pi/4$ in (59) and a shift of space and time coordinates, c_{12} can be eliminated. Then the formula for this second-order rogue wave has one real parameter c_{11} and becomes

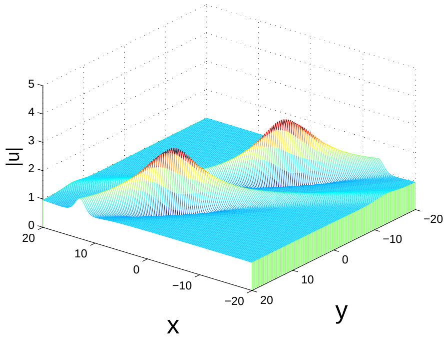
$$u[2] = -1 + \frac{(1-2ic_{11})[16(x-y)^2 - 128t] - 32i(x^2 - y^2 - c_{11} + 2c_{11}^3) + 96c_{11}^2}{4[(x-y)^2 + 8t - 2c_{11}]^2 + 32[c_{11}(x-y) + \frac{1}{2}(x+y)]^2 + 8(x-y)^2 + 16c_{11}^2}. \tag{60}$$

As long as $c_{11} \neq 0$, (60) represents a rogue wave. It starts from two lumps when $t \to -\infty$ and those two lumps move towards each other as t increases. It ultimately approaches the constant background as $t \to \infty$. This solution (60) is shown in Fig.3.

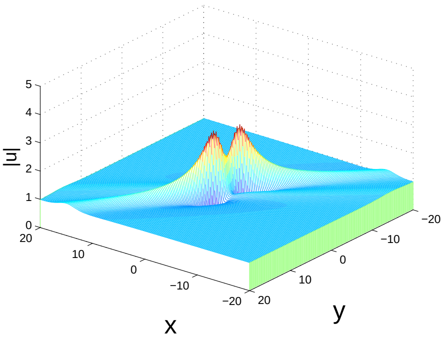
In general, the N-th order rogue wave solution is obtained by letting $\beta_k \to \beta_1, k = 2, 3, \ldots, N (N \geq 2)$ in (49). The determinant representation for N-th order rogue wave is



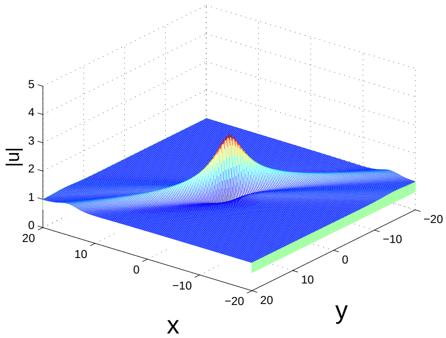
(a) $t = -10$



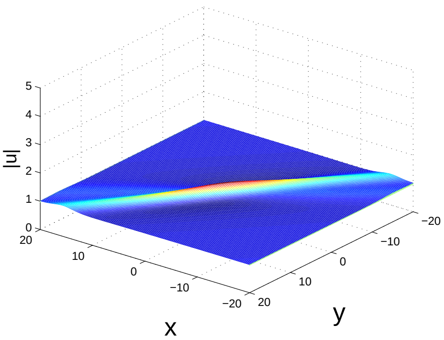
(b) $t = -5$



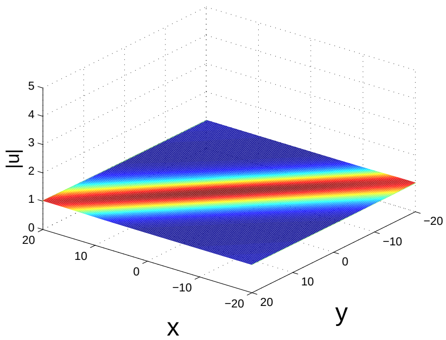
(c) $t = 0$



(d) $t = 0.5$



(e) $t = 2$



(f) $t = 20$

Figure 3: A second-order rogue wave (60) *with* $c_{11} = -1$.

$$
u[N]=u+2i\frac{\begin{vmatrix}
\partial_x^{N-1}\psi_1 & \dots & \partial_x^{N-1}\psi_1[1,N-1] & \partial_x^{N-1}\phi_1^* & \dots & \partial_x^{N-1}\phi_1[1,N-1]^* \\
\partial_x^{N-2}\psi_1 & \dots & \partial_x^{N-2}\psi_1[1,N-1] & \partial_x^{N-2}\phi_1^* & \dots & \partial_x^{N-2}\phi_1[1,N-1]^* \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\psi_1 & \dots & \psi_1[1,N-1] & \phi_1^* & \dots & \phi_1[1,N-1]^* \\
\partial_x^{N}\psi_1 & \dots & \partial_x^{N}\psi_1[1,N-1] & \partial_x^{N}\phi_1^* & \dots & \partial_x^{N}\phi_1[1,N-1]^* \\
\partial_x^{N-2}\phi_1 & \dots & \partial_x^{N-2}\phi_1[1,N-1] & -\partial_x^{N-2}\psi_1^* & \dots & -\partial_x^{N-2}\psi_1[1,N-1]^* \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\phi_1 & \dots & \phi_1[1,N-1] & -\psi_1^* & \dots & -\psi_1[1,N-1]^*
\end{vmatrix}}{\begin{vmatrix}
\partial_x^{N-1}\psi_1 & \dots & \partial_x^{N-1}\psi_1[1,N-1] & \partial_x^{N-1}\phi_1^* & \dots & \partial_x^{N-1}\phi_1[1,N-1]^* \\
\partial_x^{N-2}\psi_1 & \dots & \partial_x^{N-2}\psi_1[1,N-1] & \partial_x^{N-2}\phi_1^* & \dots & \partial_x^{N-2}\phi_1[1,N-1]^* \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\psi_1 & \dots & \psi_1[1,N-1] & \phi_1^* & \dots & \phi_1[1,N-1]^* \\
\partial_x^{N-1}\phi_1 & \dots & \partial_x^{N-1}\phi_1[1,N-1] & -\partial_x^{N-1}\psi_1^* & \dots & -\partial_x^{N-1}\psi_1[1,N-1]^* \\
\partial_x^{N-2}\phi_1 & \dots & \partial_x^{N-2}\phi_1[1,N-1] & -\partial_x^{N-2}\psi_1^* & \dots & -\partial_x^{N-2}\psi_1[1,N-1]^* \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\phi_1 & \dots & \phi_1[1,N-1] & -\psi_1^* & \dots & -\psi_1[1,N-1]^*
\end{vmatrix}}, \tag{61}
$$

where

$$
\begin{aligned}
&\begin{pmatrix} \psi_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} e^{i\beta_1/2}e^{ix\sin\beta_1+iy\cos\beta_1-t\sin 2\beta_1}[ix\cos\beta_1-iy\sin\beta_1-2t\cos 2\beta_1+i/2+f(\beta_1)] \\ ie^{-i\beta_1/2}e^{ix\sin\beta_1+iy\cos\beta_1-t\sin 2\beta_1}[ix\cos\beta_1-iy\sin\beta_1-2t\cos 2\beta_1-i/2+f(\beta_1)] \end{pmatrix}, \\
&\quad f(\beta_1)=\sum_{k=1}^{N-1}c_k\beta_1^{k-1}, \quad c_k\in\mathbb{C},
\end{aligned} \tag{62}
$$

and $\psi_1[1,k]$ and $\phi_1[1,k]$ $(1\le k\le N-1)$ are defined as in (57).

In the general N-th order rogue wave solution (61), we conjecture that β_1 needs to be fixed as $\pm\pi/4$, then c_k are free complex parameters ($1 \leq k \leq N-1, N \geq 2$). Thus it turns out that the N-th order rogue wave solution has $N-1$ reducible free complex parameters, that is $2(N-1)$ reducible free real parameters.

2.2.3 Exploding Rogue Wave Solutions

The third type of non-fundamental rogue wave solutions is the exploding rogue waves. They appear for both multi-rogue waves and higher-order rogue waves. The characteristic of this kind of rogue waves is that they blow up to infinity in certain intermediate times.

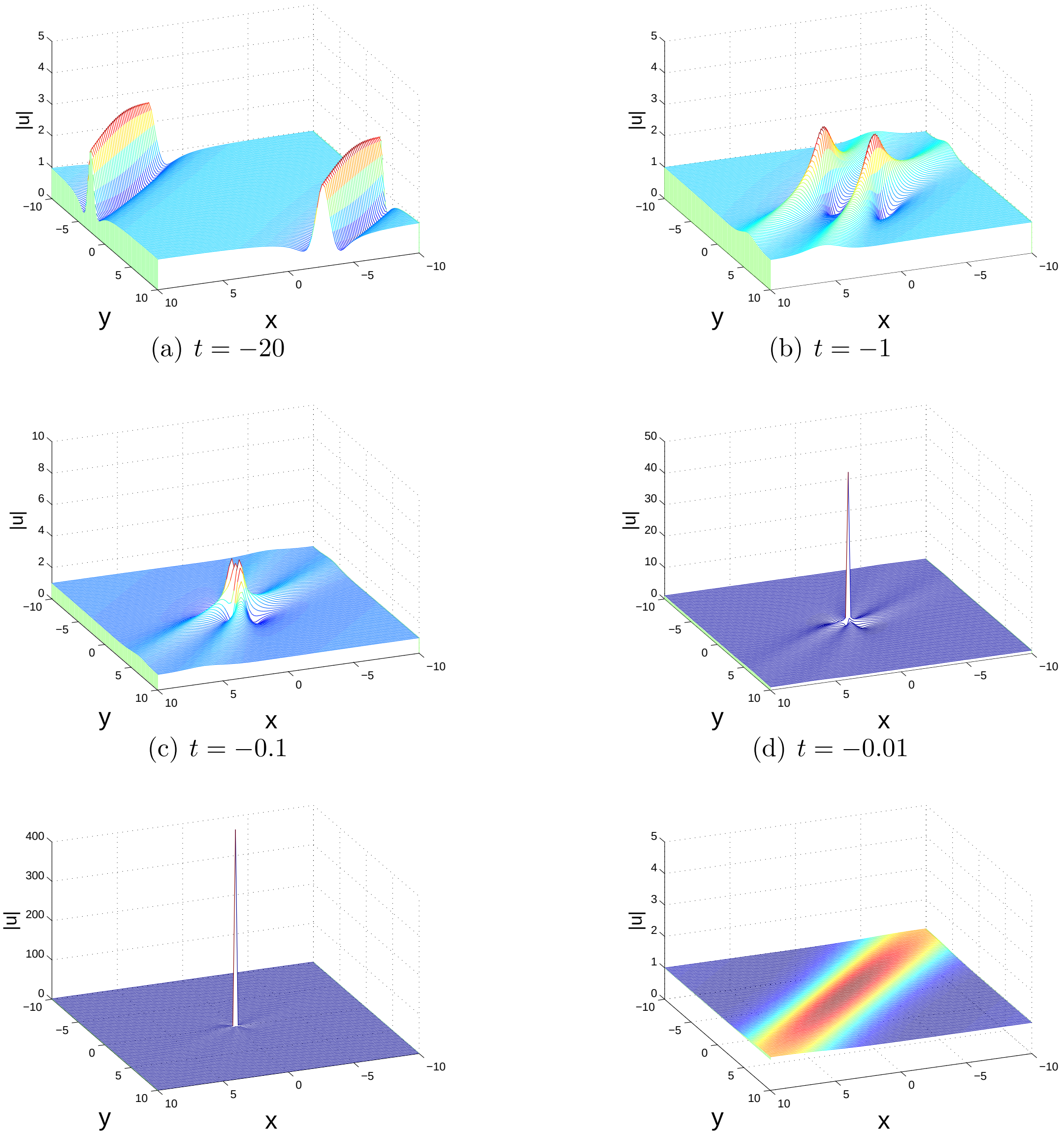
Below we will first examine the exploding rogue waves derived from the second-order rogue wave (60). Setting $c_{11} = 0$ in (60) we get

$$u[2] = -1 + \frac{16(x-y)^2 - 128t - 32i(x^2-y^2)}{4((x-y)^2+8t)^2 + 16(x^2+y^2)}. \tag{63}$$

This solution arises from two lumps, but it blows up to infinity at $t = 0$,and goes to constant background as $t \to \infty$. To illustrate, we let $(x, y) = (0, 0)$, then (63) becomes

$$u[2]|_{(x,y)=(0,0)} = -1 - \frac{1}{2t}, \tag{64}$$

and therefore this solution blows up to infinity at $t_0 = 0$. The solution is displayed in Fig.4.



(a) $t = -20$ (b) $t = -1$

(c) $t = -0.1$ (d) $t = -0.01$

(e) $t = 0.001$ (f) $t = 5$

Figure 4: An exploding second-order rogue wave (63).

Multi-rogue waves can also generate exploding rogue waves under certain parameters. For example, when we take

$$N = 2,\ \beta_1 = \pi,\ \beta_2 = \pi/2,\ c_{11} = c_{12} = c_{21} = c_{22} = 0, \tag{65}$$

we get such two-rogue wave solution

$$u[2] = \frac{-16x^2y^2 + (2x^2 + 2y^2 + 8t^2 + 5/2)(6 - 32t^2) - 64it(x^2 - y^2)}{16x^2y^2 + (16t^2 + 1)(4x^2 + 4y^2) + (16t^2 - 3)^2}, \tag{66}$$

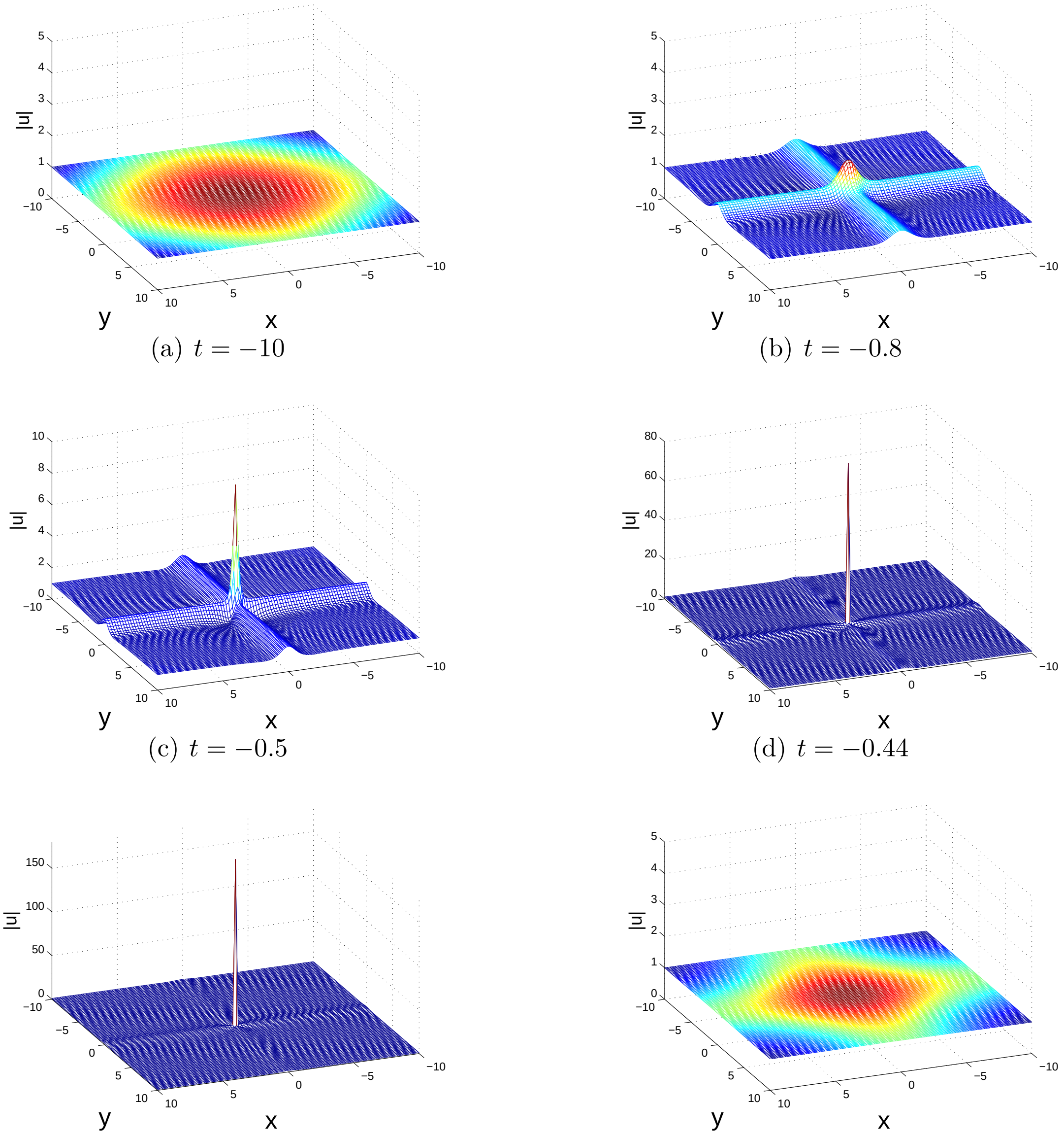
and at the origin $(x, y) = (0, 0)$,

$$u[2] = -1 + \frac{8}{3 - 16t^2}. \tag{67}$$

Obviously this wave approaches infinity when $t_0 = \pm\sqrt{3}/4$ and its graph is shown in Fig.5.

2.3 Conclusions

In conclusion, we derive three types of rogue waves for DS-II equation via Darboux transformation and this matches what have been reported in [20]. We think the most important part is to determine the solution for Lax pair (spectral function), and it is our proper choice of spectral function that leads us to the success for finding rogue waves. The idea of generalized Darboux transformation, which is in fact taking the limit of parameters, can be generalized to N case.



(a) $t = -10$ (b) $t = -0.8$

(c) $t = -0.5$ (d) $t = -0.44$

(e) $t = 0.43$ (f) $t = 5$

Figure 5: An exploding two rogue wave (66).

3 Darboux Transformation for NLS Equation

The Darboux transformation for NLS equation has been given in the monograph [25]. However, the proof for the N-fold Darboux transformation is not easy. In this section, an alternative way, which is a systematic method presented in [28, 29, 30], is applied to derive the N-fold Darboux transformation for the NLS equation. One advantage of this method is that the proof for N-fold Darboux transformation is very straightforward. At the end of this section, we show the equivalence of those two Darboux transformations.

Considering the linear problem:

$$\Psi_x = U\Psi, \quad U = \begin{pmatrix} i\lambda & ir \\ iq & -i\lambda \end{pmatrix}, \tag{68a}$$

$$\Psi_t = V\Psi, \quad V = \begin{pmatrix} 2i\lambda^2 - iqr & 2i\lambda r + r_x \\ 2i\lambda q - q_x & -2i\lambda^2 + iqr \end{pmatrix}. \tag{68b}$$

where $\Psi = (\varphi, \psi) = \begin{pmatrix} \varphi_1 & \psi_1 \\ \varphi_2 & \psi_2 \end{pmatrix}$ is a 2×2 matrix function.

The compatibility condition $\Psi_{xt} = \Psi_{tx}$ yields the zero curvature equation

$$U_t - V_x + [U, V] = 0, \tag{69}$$

where $[U, V] \equiv UV - VU$ is a commutator. By rewriting (69) in components, we arrive at the so-called nonlinear Schrödinger system

$$ir_t - r_{xx} - 2qr^2 = 0, \tag{70a}$$

$$iq_t + q_{xx} + 2q^2r = 0. \tag{70b}$$

Eq.(70) reduces to the well known nonlinear Schrödinger(NLS) equation in its self-focusing case (1).

$$r = q^*. \tag{71}$$

3.1 Darboux Transformation

The $N-$fold Darboux transformation for the nonlinear Schrödinger system (70) is a special gauge transformation[28]

$$\Psi[N] = T(x,t,\lambda)\Psi. \tag{72}$$

In order that $T(x,t,\lambda)$ is a Darboux transformation, we require (72) to satisfy the (68) with new matrices $U[N]$ and $V[N]$:

$$\Psi[N]_x = U[N]\Psi[N], \tag{73a}$$

$$\Psi[N]_t = V[N]\Psi[N], \tag{73b}$$

where

$$U[N] = (T_x + TU)T^{-1}, \tag{74a}$$

$$V[N] = (T_t + TV)T^{-1}. \tag{74b}$$

By cross differentiating (73a) and (73b), we have

$$U[N]_t - V[N]_x + [U[N], V[N]] = 0. \tag{75}$$

It is easy to see that if $U[N]$ and $V[N]$ have the same form as U and V respectively, then NLS system (70) is invariant under the gauge transformation (72). Simultaneously, the old potentials q and r in U and V will be mapped into new potentials $q[N]$ and $r[N]$ respectively. Furthermore, the NLS equation (1) is invariant under the reduction condition $r[N] = q[N]^*$.

Suppose $T(x,t,\lambda)$ is a polynomial in λ with matrix coefficients, that is

$$T(x,t,\lambda) = \begin{pmatrix} A(x,t,\lambda) & B(x,t,\lambda) \\ C(x,t,\lambda) & D(x,t,\lambda) \end{pmatrix} = \sum_{j=0}^{N} a_j(x,t)\lambda^j, \tag{76}$$

where

$$a_j = \begin{pmatrix} A_j(x,t) & B_j(x,t) \\ C_j(x,t) & D_j(x,t) \end{pmatrix} (j = 0, 1, \cdots N-1), \qquad a_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{77}$$

Since $\det T(x,t,\lambda)$ has $2N$ zeros λ_k and the coefficient of λ^N is an identity matrix, then

$$\det T(\lambda) = \prod_{k=1}^{2N} (\lambda - \lambda_k). \tag{78}$$

On the other hand,

$$\det \Psi[N](\lambda) = \det T(\lambda) \det \Psi(\lambda), \tag{79}$$

provided that $\det \Psi$ is regular at $\lambda = \lambda_k (k = 1, 2, \cdots, 2N)$, then we have

$$\det \Psi[N](\lambda_k) = 0, \qquad k = 1, 2, \cdots, 2N, \tag{80}$$

that is, the column vectors $\varphi[N] = \begin{pmatrix} \varphi_1[N] \\ \varphi_2[N] \end{pmatrix}$ and $\psi[N] = \begin{pmatrix} \psi_1[N] \\ \psi_2[N] \end{pmatrix}$ of $\Psi[N]$ are linearly dependent,

$$\varphi[N](\lambda_k) = b_k \psi[N](\lambda_k), \qquad k = 1, 2, \cdots, 2N. \tag{81}$$

Eq.(81) yield a linear system for the coefficients A_j,B_j,C_j and $D_j (j = 0, 1, \cdots, N-1)$. In fact, we express $\varphi[N](\lambda_k)$ and $\psi[N](\lambda_k)$ by transformation (72),

$$\varphi[N](\lambda_k) = \begin{pmatrix} \sum_{j=0}^{N} [A_j \varphi_1(\lambda_k) + B_j \varphi_2(\lambda_k)] \lambda_k^j \\ \sum_{j=0}^{N} [C_j \varphi_1(\lambda_k) + D_j \varphi_2(\lambda_k)] \lambda_k^j \end{pmatrix}, \tag{82a}$$

$$\psi[N](\lambda_k) = \begin{pmatrix} \sum_{j=0}^{N} [A_j \psi_1(\lambda_k) + B_j \psi_2(\lambda_k)] \lambda_k^j \\ \sum_{j=0}^{N} [C_j \psi_1(\lambda_k) + D_j \psi_2(\lambda_k)] \lambda_k^j \end{pmatrix}, \tag{82b}$$

and use (81), then we obtain the equations,

$$\sum_{j=0}^{N-1} (A_j + \beta_k B_j) \lambda_k^j = -\lambda_k^N, \tag{83a}$$

$$\sum_{j=0}^{N-1} (D_j + \alpha_k C_j) \lambda_k^j = -\lambda_k^N, \tag{83b}$$

where

$$\beta_k = \frac{1}{\alpha_k} = \frac{\varphi_2(\lambda_k) - b_k \psi_2(\lambda_k)}{\varphi_1(\lambda_k) - b_k \psi_1(\lambda_k)}. \tag{84}$$

We can solve this linear system by Cramer's rule to get the coefficients A_j, B_j, C_j and $D_j (j = 0, 1, \cdots, N-1)$ and hence A, B, C and D.

Next we are going to show that $U[N]$ and $V[N]$ have the same form of U and V.

Theorem 2. *The matrix $U[N]$ and $V[N]$ determined by (74) have the form*

$$U[N] = \begin{pmatrix} i\lambda & ir[N] \\ iq[N] & -i\lambda \end{pmatrix}, \tag{85a}$$

$$V[N] = \begin{pmatrix} 2i\lambda^2 - iq[N]r[N] & 2i\lambda r[N] + r[N]_x \\ 2i\lambda q[N] - q[N]_x & -2i\lambda^2 + iq[N]r[N] \end{pmatrix}, \tag{85b}$$

where

$$q[N] = q + 2C_{N-1}, \qquad r[N] = r - 2B_{N-1}. \tag{86}$$

Proof. Since $T^{-1} = \dfrac{T^*}{\det T}$, and $T^* = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$ is the adjugate matrix of T, and suppose

$$(T_x + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}, \tag{87}$$

then

$$f_{11}(\lambda) = A_xD - B_xC + i\lambda(AD + BC) + iqBD - irAC, \tag{88a}$$

$$f_{12}(\lambda) = -A_x + B_xA - 2i\lambda AB - iqB^2 + irA^2, \tag{88b}$$

$$f_{21}(\lambda) = -D_xC + C_xD + 2i\lambda CD + iqD^2 - irC^2, \tag{88c}$$

$$f_{22}(\lambda) = -C_xB + D_xA - i\lambda(AD + BC) - iqBD + irAC. \tag{88d}$$

Since $A = \lambda^N I + \sum_{j=0}^{N-1} A_j\lambda^j$, $B = \sum_{j=0}^{N-1} B_j\lambda^j$, $C = \sum_{j=0}^{N-1} C_j\lambda^j$ and $D = \lambda^N I + \sum_{j=0}^{N-1} D_j\lambda^j$, it is obvious that $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are $(2N+1)$th order polynomial, and $f_{12}(\lambda)$ and $f_{21}(\lambda)$ are $2N$th order polynomial.

In addition, it follows from the linear system of equations (83) that

$$A(\lambda_k) = -\beta_k B(\lambda_k), \tag{89a}$$

$$C(\lambda_k) = -\beta_k D(\lambda_k), \qquad k = 1, 2, \cdots, 2N. \tag{89b}$$

Also,

$$\beta_{k,x} = iq - 2i\lambda_k\beta_k - ir\beta_k^2. \tag{90}$$

Then we can verify that all zeros $\lambda_k (k = 1, 2, \cdots, 2N)$ of $\det T$ are roots of $f_{ij}(\lambda)(i, j = 1, 2)$. Here we only give the verification for $f_{11}(\lambda)$ and the verifications for others are very similar.

$$\begin{aligned} f_{11}(\lambda_k) &= A_x D - B_x C + i\lambda(AD + BC) + iqBD - irAC|_{\lambda=\lambda_k} \\ &= [-\beta_{k,x}B(\lambda_k) - \beta_k B_x(\lambda_k)]\, D(\lambda_k) + \beta_k B_x(\lambda_k) D(\lambda_k) - 2i\beta_k\lambda_k B(\lambda_k) D(\lambda_k) \\ &+ iqB(\lambda_k)D(\lambda_k) - ir\beta_k^2 B(\lambda_k)D(\lambda_k) \\ &= [\beta_{k,x} - 2i\beta_k\lambda_k + iq - ir\beta_k^2]B(\lambda_k)D(\lambda_k) \\ &= 0. \end{aligned} \tag{91}$$

Since $\lambda_k (k = 1, 2, \cdots, 2N)$ are roots of $f_{ij}(\lambda)(i, j = 1, 2)$, then matrix $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ can be divided by $\det T$, i.e. $(T_x + TU)T^*$ can be divided by $\det T$, thus $(T_x + TU)T^{-1}$, which is $U[N]$, is a linear function of λ. We can suppose

$$U[N] = \widetilde{U_0}(x, t) + \widetilde{U_1}(x, t)\lambda, \tag{92}$$

where matrices $\widetilde{U_0}(x, t)$ and $\widetilde{U_1}(x, t)$ do not depend on λ, then

$$(T_x + TU)T^{-1} = U[N] = \widetilde{U_0}(x, t) + \widetilde{U_1}(x, t)\lambda. \tag{93}$$

We denote $U = U_0(x,t) + U_1(x,t)\lambda$ with

$$U_0 = \begin{pmatrix} 0 & ir \\ iq & 0 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{94}$$

Comparing the coefficients of λ^{N+1} and λ^N in $T_x + TU = [\widetilde{U_0}(x,t) + \widetilde{U_1}(x,t)\lambda]T$, we have

$$\lambda^{N+1}: \qquad a_N U_1 = \widetilde{U_1} a_N, \tag{95a}$$

$$\lambda^N: \qquad U_0 + a_{N-1}U_1 = \widetilde{U_0} + \widetilde{U_1} a_{N-1}. \tag{95b}$$

Thus

$$\widetilde{U_1} = U_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{96a}$$

$$\widetilde{U_0} = U_0 + a_{N-1}U_1 - \widetilde{U_1}a_{N-1} = \begin{pmatrix} 0 & ir - 2iB_{N-1} \\ iq + 2iC_{N-1} & 0 \end{pmatrix}. \tag{96b}$$

and

$$U[N] = \begin{pmatrix} i\lambda & ir[N] \\ iq[N] & -i\lambda \end{pmatrix} \tag{97}$$

with $q[N]$ and $r[N]$ defined in (86). We finish the proof for spatial part. Next we will prove that $V[N]$ has the same form of V as well.

In a way similar to the previous proof, we can verify that $(T_t + TV)T^{-1}$ is a quadratic function in λ with matrix coefficients. To do so, we first assume that

$$(T_t + TV)T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}, \tag{98}$$

with

$$g_{11}(\lambda) = A_tD - B_tC + (2i\lambda^2 - iqr)(AD + BC) + 2i\lambda(qBD + rAC) - q_xBD - r_xAC, \tag{99a}$$

$$g_{12}(\lambda) = AB_t - A_tB - 2(2i\lambda^2 - iqr)AB + 2i\lambda(rA^2 - qB^2) + q_xB^2 + r_xA^2, \tag{99b}$$

$$g_{21}(\lambda) = DC_t - D_tC + 2(2i\lambda^2 - iqr)CD + 2i\lambda(qD^2 - rC^2) - q_xD^2 - r_xC^2, \tag{99c}$$

$$g_{22}(\lambda) = D_tA - BC_t - (2i\lambda^2 - iqr)(AD + BC) - 2i\lambda(qBD - rAC) + q_xBD + r_xAC. \tag{99d}$$

We find that $g_{11}(\lambda)$ and $g_{22}(\lambda)$ are $(2N+2)$th order polynomial in λ and $g_{12}(\lambda)$ and $g_{21}(\lambda)$ are $2N$th order polynomial. Next we can show that all zeros $\lambda_k(k = 1, 2, \cdots, 2N)$ are roots of $g_{ij}(\lambda)(i, j = 1, 2)$. This time we would like to justify g_{12} for example. Before that, we point out that

$$\beta_{k,t} = 2i\lambda_k q - q_x - 2(2i\lambda_k^2 - iqr)\beta_k - (2i\lambda_k r + r_x)\beta_k^2. \tag{100}$$

In fact,

$$\begin{aligned}
g_{21}(\lambda_k) &= \left[DC_t - D_tC + 2(2i\lambda^2 - iqr)CD + 2i\lambda(qD^2 - rC^2) - q_xD^2 - r_xC^2\right]|_{\lambda=\lambda_k} \\
&= D(\lambda_k)\left[-\beta_{k,t} - \beta_k D_t(\lambda_k)\right] + \beta_k D_t(\lambda_k)D(\lambda_k) - 2\beta_k(2i\lambda_k^2 - iqr)D^2(\lambda_k) \\
&+ 2i\lambda_k(qD^2(\lambda_k) - r\beta_k^2 D^2(\lambda_k)) - q_xD^2(\lambda_k) - \beta_k^2 r_x D^2(\lambda_k)) \\
&= \left[-\beta_{k,t} - 2(2i\lambda_k^2 - iqr)\beta_k + 2i\lambda_k q - q_x - (2i\lambda_k r + r_x)\beta_k^2\right] D^2(\lambda_k) \\
&= 0.
\end{aligned} \tag{101}$$

Since $\lambda_k(k = 1, 2, \cdots, 2N)$ are roots of $g_{ij}(\lambda)(i, j = 1, 2)$, then matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ can be divided by $\det T$, i.e. $(T_t + TV)T^*$ can be divided by $\det T$, thus $(T_t + TV)T^{-1}$, which is $V[N]$, is a quadratic function of λ, that is

$$(T_T + TV)T^{-1} = V[N] = \widetilde{V_0}(x,t) + \widetilde{V_1}(x,t)\lambda + \widetilde{V_2}(x,t)\lambda^2, \tag{102}$$

where matrices $\widetilde{V_0}(x,t)$, $\widetilde{V_1}(x,t)$ and $\widetilde{V_2}(x,t)$ do not depend on λ. Besides, we denote $V = V_0(x,t) + V_1(x,t)\lambda + V_2(x,t)\lambda^2$ with

$$V_0 = \begin{pmatrix} -iqr & r_x \\ -q_x & iqr \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 0 & 2ir \\ 2iq & 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \tag{103}$$

Comparing the coefficients of $\lambda^{N+j}(j = 0,1,2)$ in $T_t + TV = [\widetilde{V_0}(x,t) + \widetilde{V_1}(x,t)\lambda + \widetilde{V_2}(x,t)\lambda^2]T$ yields

$$\lambda^{N+2}: \qquad a_N V_2 = \widetilde{V_2} a_N, \tag{104a}$$

$$\lambda^{N+1}: \qquad a_N V_1 + a_{N-1} V_2 = \widetilde{V_1} a_N + \widetilde{V_2} a_{N-1}, \tag{104b}$$

$$\lambda^{N}: \quad a_N V_0 + a_{N-1} V_1 + a_{N-2} V_2 = \widetilde{V_0} a_N + \widetilde{V_1} a_{N-1} + \widetilde{V_2} a_{N-2}, \tag{104c}$$

Thus

$$\widetilde{V_2} = V_2 = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}, \tag{105a}$$

$$\widetilde{V_1} = V_1 + a_{N-1}V_2 - \widetilde{V_2} a_{N-1} = \begin{pmatrix} 0 & 2ir[N] \\ 2iq[N] & 0 \end{pmatrix}, \tag{105b}$$

$$\widetilde{V_0} = V_0 + a_{N-1}V_1 - \widetilde{V_1} a_{N-1} + a_{N-2}V_2 - \widetilde{V_2} a_{N-2}. \tag{105c}$$

Comparing the coefficient of λ^{N-1} in $T_x + TU = U[1]T$, we have

$$a_{N-1,x} = -\left(a_{N-1}U_0 - \widetilde{U_0} a_{N-1} + a_{N-2}U_1 - \widetilde{U_1} a_{N-2}\right), \tag{106}$$

and notice that

$$U_0 = \frac{1}{2}V_1, \quad \widetilde{U_0} = \frac{1}{2}\widetilde{V_1}, \quad U_1 = \frac{1}{2}V_2, \quad \widetilde{U_1} = \frac{1}{2}\widetilde{V_2}. \tag{107}$$

We find

$$\widetilde{V_0} = V_0 - 2a_{N-1,x}, \tag{108}$$

then direct calculation shows that

$$\widetilde{V_0} = \begin{pmatrix} -iq[N]r[N] & r[N]_x \\ q[N]_x & iq[N]r[N] \end{pmatrix}. \tag{109}$$

This completes the proof. $\square$

According to Theorem 2, the transformation (72) and (86) transform the Lax pair (68) into another Lax pair (73) of the same type. Therefore, both Lax pair yield the NLS system (70). The transformation

$$(\Psi, q, r) \mapsto (\Psi[N], q[N], r[N]) \tag{110}$$

is called a $N-$fold Darboux transformation of NLS system (70).

3.2 Reduction of Darboux Transformation

In this section, we will consider the transformed potentials $r[N]$ and $q[N]$ of NLS system, then use the reduction condition $r[N] = q[N]^*$ to get the solutions for NLS equation.

Due to Theorem 2, we know that $r[N]$ and $q[N]$ are given in terms of B_{N-1} and C_{N-1} respectively, thus we need to solve the linear system (83) for them. By writing (83) into matrix form,

$$\begin{pmatrix} 1 & \beta_1 & \lambda_1 & \lambda_1\beta_1 & \cdots & \lambda_1^{N-1} & \lambda_1^{N-1}\beta_1 \\ 1 & \beta_2 & \lambda_2 & \lambda_2\beta_2 & \cdots & \lambda_2^{N-1} & \lambda_2^{N-1}\beta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta_{2N} & \lambda_{2N} & \lambda_{2N}\beta_{2N} & \cdots & \lambda_{2N}^{N-1} & \lambda_{2N}^{N-1}\beta_{2N} \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \\ \vdots \\ B_{N-1} \end{pmatrix} = \begin{pmatrix} -\lambda_1^N \\ -\lambda_2^N \\ \vdots \\ -\lambda_{2N}^N \end{pmatrix}, \quad (111a)$$

$$\begin{pmatrix} 1 & \alpha_1 & \lambda_1 & \lambda_1\alpha_1 & \cdots & \lambda_1^{N-1} & \lambda_1^{N-1}\alpha_1 \\ 1 & \alpha_2 & \lambda_2 & \lambda_2\alpha_2 & \cdots & \lambda_2^{N-1} & \lambda_2^{N-1}\alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_{2N} & \lambda_{2N} & \lambda_{2N}\alpha_{2N} & \cdots & \lambda_{2N}^{N-1} & \lambda_{2N}^{N-1}\alpha_{2N} \end{pmatrix} \begin{pmatrix} D_0 \\ C_0 \\ \vdots \\ C_{N-1} \end{pmatrix} = \begin{pmatrix} -\lambda_1^N \\ -\lambda_2^N \\ \vdots \\ -\lambda_{2N}^N \end{pmatrix}, \quad (111b)$$

and use Cramer's rule, we get

$$r[N] = r + 2\frac{\begin{vmatrix} 1 & \beta_1 & \lambda_1 & \lambda_1\beta_1 & \cdots & \lambda_1^{N-1} & \lambda_1^N \\ 1 & \beta_2 & \lambda_2 & \lambda_2\beta_2 & \cdots & \lambda_2^{N-1} & \lambda_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta_{2N} & \lambda_{2N} & \lambda_{2N}\beta_{2N} & \cdots & \lambda_{2N}^{N-1} & \lambda_{2N}^N \end{vmatrix}}{\begin{vmatrix} 1 & \beta_1 & \lambda_1 & \lambda_1\beta_1 & \cdots & \lambda_1^{N-1} & \lambda_1^{N-1}\beta_1 \\ 1 & \beta_2 & \lambda_2 & \lambda_2\beta_2 & \cdots & \lambda_2^{N-1} & \lambda_2^{N-1}\beta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta_{2N} & \lambda_{2N} & \lambda_{2N}\beta_{2N} & \cdots & \lambda_{2N}^{N-1} & \lambda_{2N}^{N-1}\beta_{2N} \end{vmatrix}}, \tag{112a}$$

$$q[N] = q - 2\frac{\begin{vmatrix} 1 & \alpha_1 & \lambda_1 & \lambda_1\alpha_1 & \cdots & \lambda_1^{N-1} & \lambda_1^N \\ 1 & \alpha_2 & \lambda_2 & \lambda_2\alpha_2 & \cdots & \lambda_2^{N-1} & \lambda_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_{2N} & \lambda_{2N} & \lambda_{2N}\alpha_{2N} & \cdots & \lambda_{2N}^{N-1} & \lambda_{2N}^N \end{vmatrix}}{\begin{vmatrix} 1 & \alpha_1 & \lambda_1 & \lambda_1\alpha_1 & \cdots & \lambda_1^{N-1} & \lambda_1^{N-1}\alpha_1 \\ 1 & \alpha_2 & \lambda_2 & \lambda_2\alpha_2 & \cdots & \lambda_2^{N-1} & \lambda_2^{N-1}\alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_{2N} & \lambda_{2N} & \lambda_{2N}\alpha_{2N} & \cdots & \lambda_{2N}^{N-1} & \lambda_{2N}^{N-1}\alpha_{2N} \end{vmatrix}}. \tag{112b}$$

The reduction condition $r[N] = q[N]^*$ will be realized when

$$\lambda_{2j} = \lambda_{2j-1}^*, \qquad \alpha_{2j} = -\frac{1}{\alpha_{2j-1}^*}, \qquad \beta_{2j} = -\frac{1}{\beta_{2j-1}^*}. \tag{113}$$

The solution of NLS equation is given by

$$q[N] = q - 2\frac{\begin{vmatrix} 1 & \alpha_1 & \lambda_1 & \lambda_1\alpha_1 & \cdots & \lambda_1^{N-1} & \lambda_1^{N} \\ 1 & -\alpha_1^{-1*} & \lambda_1^* & -\lambda_1^*\alpha_1^{-1*} & \cdots & \lambda_1^{*N-1} & \lambda_1^{*N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -\alpha_N^{-1*} & \lambda_N^* & -\lambda_N^*\alpha_N^{-1*} & \cdots & \lambda_N^{*N-1} & \lambda_N^{*N} \end{vmatrix}}{\begin{vmatrix} 1 & \alpha_1 & \lambda_1 & \lambda_1\alpha_1 & \cdots & \lambda_1^{N-1} & \lambda_1^{N-1}\alpha_1 \\ 1 & -\alpha_1^{-1*} & \lambda_1^* & -\lambda_1^*\alpha_1^{-1*} & \cdots & \lambda_1^{*N-1} & -\lambda_1^{*N-1}\alpha_1^{-1*} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -\alpha_N^{-1*} & \lambda_N^* & -\lambda_N^*\alpha_N^{-1*} & \cdots & \lambda_N^{*N-1} & -\lambda_N^{*N-1}\alpha_N^{-1*} \end{vmatrix}}, \tag{114}$$

where

$$\alpha_k = \frac{\varphi_1(\lambda_k) - b_k\psi_1(\lambda_k)}{\varphi_2(\lambda_k) - b_k\psi_2(\lambda_k)}. \tag{115}$$

3.3 The Equivalence of Two Methods

In this section we will show that the transformed solution we get here coincides with the one in the literature [25]. We will take $N = 2$ as an example.

Since $\begin{pmatrix} \varphi_1(\lambda_k) & \psi_1(\lambda_k) \\ \varphi_2(\lambda_k) & \psi_2(\lambda_k) \end{pmatrix}$ is a matrix solution of lax pair (68) at $\lambda = \lambda_k$, then $\begin{pmatrix} \varphi_1(\lambda_k) & \psi_1(\lambda_k) \\ \varphi_2(\lambda_k) & \psi_2(\lambda_k) \end{pmatrix}\begin{pmatrix} 1 \\ -b_k \end{pmatrix}$ is a column solution of it at $\lambda = \lambda_k$. If we use the notation in [25], we have

$$\begin{pmatrix} \varphi_1(\lambda_k) & \psi_1(\lambda_k) \\ \varphi_2(\lambda_k) & \psi_2(\lambda_k) \end{pmatrix}\begin{pmatrix} 1 \\ -b_k \end{pmatrix} = \begin{pmatrix} \varphi_1(\lambda_k) - b_k\psi_1(\lambda_k) \\ \varphi_2(\lambda_k) - b_k\psi_2(\lambda_k) \end{pmatrix} = \begin{pmatrix} \psi_k \\ \phi_k \end{pmatrix}, \tag{116}$$

then

$$\alpha_k = \frac{\psi_k}{\phi_k}, \qquad \alpha_k^{-1*} = -\frac{\phi_k^*}{\psi_k^*}. \tag{117}$$

Hence

$$
\begin{aligned}
q[2] &= q - 2\frac{\begin{vmatrix} 1 & \psi_1/\phi_1 & \lambda_1 & \lambda_1^2 \\ 1 & -\phi_1^*/\psi_1^* & \lambda_1^* & \lambda_1^{*2} \\ 1 & \psi_2/\phi_2 & \lambda_2 & \lambda_2^2 \\ 1 & -\phi_2^*/\psi_2^* & \lambda_2^* & \lambda_2^{*2} \end{vmatrix}}{\begin{vmatrix} 1 & \psi_1/\phi_1 & \lambda_1 & \lambda_1\psi_1/\phi_1 \\ 1 & -\phi_1^*/\psi_1^* & \lambda_1^* & -\lambda_1^*\phi_1^*/\psi_1^* \\ 1 & \psi_2/\phi_2 & \lambda_2 & \lambda_2\psi_2/\phi_2 \\ 1 & -\phi_2^*/\psi_2^* & \lambda_2^* & -\lambda_2\phi_2^*/\psi_2^* \end{vmatrix}} = q - 2\frac{\begin{vmatrix} \phi_1 & \psi_1 & \lambda_1\phi_1 & \lambda_1^2\phi_1 \\ \psi_1^* & -\phi_1^* & \lambda_1^*\psi_1^* & \lambda_1^{*2}\psi_1^* \\ \phi_2 & \psi_2 & \lambda_2\phi_2 & \lambda_2^2\phi_2 \\ \psi_2^* & -\phi_2^* & \lambda_2^*\psi_2^* & \lambda_2^{*2}\psi_2^* \end{vmatrix}}{\begin{vmatrix} \phi_1 & \psi_1 & \lambda_1\phi_1 & \lambda_1\psi_1 \\ \psi_1^* & -\phi_1^* & \lambda_1^*\psi_1^* & -\lambda_1^*\phi_1^* \\ \phi_2 & \psi_2 & \lambda_2\phi_2 & \lambda_2\psi_2 \\ \psi_2^* & -\phi_2^* & \lambda_2^*\psi_2^* & -\lambda_2\phi_2^* \end{vmatrix}} \\
&= q - 2\frac{\begin{vmatrix} \phi_1 & \psi_1 & \lambda_1\phi_1 & \lambda_1^2\phi_1 \\ \phi_2 & \psi_2 & \lambda_2\phi_2 & \lambda_2^2\phi_2 \\ \psi_1^* & -\phi_1^* & \lambda_1^*\psi_1^* & \lambda_1^{*2}\psi_1^* \\ \psi_2^* & -\phi_2^* & \lambda_2^*\psi_2^* & \lambda_2^{*2}\psi_2^* \end{vmatrix}}{\begin{vmatrix} \phi_1 & \psi_1 & \lambda_1\phi_1 & \lambda_1\psi_1 \\ \phi_2 & \psi_2 & \lambda_2\phi_2 & \lambda_2\psi_2 \\ \psi_1^* & -\phi_1^* & \lambda_1^*\psi_1^* & -\lambda_1^*\phi_1^* \\ \psi_2^* & -\phi_2^* & \lambda_2^*\psi_2^* & -\lambda_2\phi_2^* \end{vmatrix}} = q - 2\frac{\begin{vmatrix} \phi_1 & \lambda_1\phi_1 & \psi_1 & \lambda_1^2\phi_1 \\ \phi_2 & \lambda_2\phi_2 & \psi_2 & \lambda_2^2\phi_2 \\ \psi_1^* & \lambda_1^*\psi_1^* & -\phi_1^* & \lambda_1^{*2}\psi_1^* \\ \psi_2^* & \lambda_2^*\psi_2^* & -\phi_2^* & \lambda_2^{*2}\psi_2^* \end{vmatrix}}{\begin{vmatrix} \phi_1 & \lambda_1\phi_1 & \psi_1 & \lambda_1\psi_1 \\ \phi_2 & \lambda_2\phi_2 & \psi_2 & \lambda_2\psi_2 \\ \psi_1^* & \lambda_1^*\psi_1^* & -\phi_1^* & -\lambda_1^*\phi_1^* \\ \psi_2^* & \lambda_2^*\psi_2^* & -\phi_2^* & -\lambda_2\phi_2^* \end{vmatrix}} \qquad (118) \\
&= q - 2\frac{\begin{vmatrix} \lambda_1^2\phi_1 & \psi_1 & \lambda_1\phi_1 & \phi_1 \\ \lambda_2^2\phi_2 & \psi_2 & \lambda_2\phi_2 & \phi_2 \\ \lambda_1^{*2}\psi_1^* & -\phi_1^* & \lambda_1^*\psi_1^* & \psi_1^* \\ \lambda_2^{*2}\psi_2^* & -\phi_2^* & \lambda_2^*\psi_2^* & \psi_2^* \end{vmatrix}}{\begin{vmatrix} \lambda_1\psi_1 & \psi_1 & \lambda_1\phi_1 & \phi_1 \\ \lambda_2\psi_2 & \psi_2 & \lambda_2\phi_2 & \phi_2 \\ -\lambda_1^*\phi_1^* & -\phi_1^* & \lambda_1^*\psi_1^* & \psi_1^* \\ -\lambda_2\phi_2^* & -\phi_2^* & \lambda_2^*\psi_2^* & \psi_2^* \end{vmatrix}}.
\end{aligned}
$$

The last equation is exactly the solution formula given in [11, 25] for $N = 2$.

Two-fold Darboux transformation (118) can be used to find the two soliton solution [25] and second order rogue wave soltuion [11] with different choice of seed solution and spectral function. More details for calculating those solutions can be found in [11, 25].

3.4 Conclusions

In conclusion, we use an alternative way to derive the Darboux transformation, with which the proof can be easily done. We also show that the Darboux transformation we derive here is equivalent to the one in literature [11, 25].

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